1) a) $\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v, v+w\rangle + \langle w, v+w\rangle$
$= \langle v+w, v\rangle + \langle v+w, w\rangle = \langle v,v\rangle + \langle w, v\rangle + \langle v, w\rangle$
$+ \langle w, w\rangle$.

Note that $\langle v, w\rangle = 0$ and $\langle w, v\rangle = \langle v, w\rangle = 0$.
Also, $\langle v, v\rangle = \|v\|^2$ and $\langle w, w\rangle = \|w\|^2$.
This completes the proof.

b) Obviously, $v = v-p+p$ and $v-p$ is orthogonal to p. Hence by part (a) we have
$\|v\|^2 = \|v-p+p\|^2 = \|v-p\|^2 + \|p\|^2$

Hence $\|v\|^2 \geq \|p\|^2$ with equality if and only if $v=p$.

We conclude that $\|p\| \leq \|v\|$ and that equality holds if and only if $v \in S$.

c) As in (b), $\|v\|^2 = \|v-p\|^2 + \|p\|^2$. Hence
$\|v\|^2 \geq \|v-p\|^2$ with equality if and only if $p=0$.

We conclude that $\|v-p\| \leq \|v\|$ with equality if and only if v is orthogonal to S

2) (a) $\langle 1, x \rangle = \int_{-1}^{1} x \, dx = 0$ since x is odd

(b) $\| 1 \|^2 = \int_{-1}^{1} dx = 2 \Rightarrow \| 1 \| = \sqrt{2}$

$\| x \| = \int_{-1}^{1} x^2 dx = \left[\frac{1}{3} x^3 \right]_{-1}^{1} = \frac{2}{3}$

$\Rightarrow \| x \| = \sqrt{2}/\sqrt{3}$

(c) $u_1(x) := \frac{1}{\sqrt{2}}$ has norm 1

$u_2(x) := \frac{\sqrt{3}}{\sqrt{2}} x$ has norm 1

$\langle u_1, u_2 \rangle = \frac{1}{2}\sqrt{3} \langle 1, x \rangle = 0$

Hence $\{u_1, u_2\}$ is an orthonormal set. Finally, every element $g(x) = a + bx$ can be written as linear combination of $u_1(x)$ and $u_2(x)$:

$$a + bx = \sqrt{2}\, a \cdot u_1(x) + \frac{\sqrt{2}}{\sqrt{3}} b \cdot u_2(x).$$

(d) Let $p(x)$ be the orthogonal projection of $g(x) = x^{1/3}$ onto S. This is the best least squares approximation. It is given by

$$p(x) = \langle g, u_1 \rangle u_1(x) + \langle g, u_2 \rangle \cdot u_2(x)$$

We compute: $\langle g, u_1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x^{1/3} dx =$

$\frac{1}{\sqrt{2}} \left[\frac{3}{4} x^{4/3} \right]_{-1}^{1} = \frac{1}{\sqrt{2}} \frac{3}{4} (1 - 1) = 0$

$$\langle f, u_2 \rangle = \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} x^{4/3} dx = \frac{\sqrt{3}}{\sqrt{2}} \frac{3}{7} \left[x^{7/3} \right]_{-1}^{1}$$

Note : $x^{7/3} = x^2 . x^{1/3}$ dus $1^{7/3} = 1$

$(-1)^{7/3} = -1$.

Conclusie : $\langle f, u_2 \rangle = \frac{\sqrt{3}}{\sqrt{2}} \frac{3}{7} . 2$

$= \frac{6\sqrt{3}}{7\sqrt{2}}$

We find:

$$p(x) = \frac{6\sqrt{3}}{7\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} x = \frac{18}{14} x = \frac{9}{7} x.$$

PJJJ....

3) a) The eigenvalues are the complex roots of the polynomial $p(\lambda) = \det(A - \lambda I)$
Hence $p(\lambda)$ can be factored as the product

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots\cdots (\lambda_n - \lambda)$$

Now compute the value of $p(\lambda)$ at $\lambda = 0$. This is equal to

$$\det(A) = p(0) = \lambda_1 \lambda_2 \cdots \lambda_n .$$

b) If all eigenvalues are nonzero then also their product $\lambda_1 \lambda_2 \cdots \lambda_n$ is nonzero. Thus $\det(A) \neq 0$ so A^{-1} exists.

c) A^{-1} is diagonalizable if and only if A is

($\Leftarrow$) A diagonalizable. There exists a nonsingular X so that $X^{-1} A X = \Lambda$ with

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \text{diagonal}$$

Hence $A = X \Lambda X^{-1}$ so $A^{-1} = (X \Lambda X^{-1})^{-1}$
$= (X^{-1})^{-1} \Lambda^{-1} X^{-1} = X \Lambda^{-1} X^{-1}$, and therefore

$$X^{-1} A^{-1} X = \Lambda^{-1}$$

X^{-1} is of course nonsingular, $\Lambda^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{pmatrix}$

diagonal.

($\Rightarrow$) We have shown that if a matrix A is diagonalizable, then A^{-1} is.
Hence, if A^{-1} is diagonalizable then $(A^{-1})^{-1}$ is, so A is.

(d) If A is unitarily diagonalizable there exists unitary U s.t. $U^H A U = \Lambda$, so $A = U \Lambda U^H$ and $A^{-1} = (U \Lambda U^H)^{-1}$
$= (U^H)^{-1} \Lambda^{-1} U^{-1} = U \Lambda^{-1} U^H$. We find $U^H A^{-1} U = \Lambda^{-1}$, so A^{-1} is unitarily diagonalizable.
Of course also the converse holds.

4) a) Assume A Hermitian. It is a theorem in the book that A has only real eigenvalues.
Also A is normal:

$$A^H . A = A . A = A . A^H \quad \text{since } A^H = A$$

b) If A is normal then by a theorem in the book it is unitarily diagonalizable, so there exists unitary U and diagonal matrix Λ s.t. $U^H A U = \Lambda$. The diagonal elements of Λ are the (real) eigenvalues.
We have $A = U \Lambda U^H$ so $A^H = (U \Lambda U^H)^H$
$= U \Lambda^H U^H = U \Lambda U^H = A$. $\Rightarrow$ A Hermitian

c) $A^H = -A$. We show it has only eigenvalues λ with $Re(\lambda) = 0$.

Let λ be an eigenvalue of A, $Ax = \lambda x$ with $x \neq 0$.

We then have $\overline{x^H A x} = (x^H A x)^H = x^H A^H x$
$= -x^H A x$.

In other words: the complex conjugate of $x^H A x$ is $-x^H A x$, so $x^H A x$ lies on the imaginary axis: $Re(x^H A x) = 0$.

Also $x^H A x = x^H \lambda x = \lambda \|x\|^2$. So

$$\lambda = \frac{x^H A x}{\|x\|^2} \quad \text{and hence} \quad Re(\lambda) = 0$$

A is also normal:

$$A^H A = -A \cdot A = A \cdot -A = A A^H.$$

d) A is normal so unitarily diagonalizable: $\exists$ unitary U s.t. $U^H A U = \Lambda$. The diagonal elements of Λ are the (purely imaginary) eigenvalues of A. Note that

$$\Lambda^H = -\Lambda$$

Hence $A^H = (U \Lambda U^H)^H = U \Lambda^H U^H = -U \Lambda U^H$
$= -A$ so A is skew Hermitian